

# Conformal invariance of the planar $\beta$ -deformed $\mathcal{N} = 4$ SYM theory requires $\beta$ real

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**ABSTRACT:** We study the  $\mathcal{N} = 1$   $SU(N)$  SYM theory which is a marginal deformation of the  $\mathcal{N} = 4$  theory, with a complex deformation parameter  $\beta$ . We consider the large  $N$  limit and study perturbatively the conformal invariance condition. We find that finiteness requires reality of the deformation parameter  $\beta$ .

**KEYWORDS:** Conformal and W Symmetry, Superspaces, AdS-CFT Correspondence.

The  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory offers one of the best playgrounds to test new ideas connected to nonperturbative and exact results. Using the AdS/CFT correspondence [1] it has allowed to get new insights and a deeper understanding of duality properties enjoyed by the gauge theory and the corresponding supergravity. The search for theories with a less degree of supersymmetry that nonetheless might possess features similar to the ones of  $\mathcal{N} = 4$  SYM has lead to consider theories obtained deforming the  $\mathcal{N} = 4$  theory itself. Of special interest are  $\mathcal{N} = 4$  marginal deformations analyzed in [2] for which the supergravity dual description has been found in [3].

In this paper we consider such marginal deformations. They are called  $\beta$ -deformations since they are obtained by modifying the original  $\mathcal{N} = 4$  superpotential for the chiral superfields in the following way

$$ig \operatorname{Tr}(\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_2) \longrightarrow ih \operatorname{Tr}\left(e^{i\pi\beta} \Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta} \Phi_1\Phi_3\Phi_2\right) \quad (1)$$

where in general  $h$  and  $\beta$  are complex constants. In [2] it was argued that these  $\beta$ -deformed  $\mathcal{N} = 1$  theories become conformally invariant, i.e. the deformation becomes exactly marginal, if one condition is satisfied by the constants  $h$  and  $\beta$ . For the case of  $\beta$  real and in the planar limit it has been shown [4] that the condition

$$h\bar{h} = g^2 \quad (2)$$

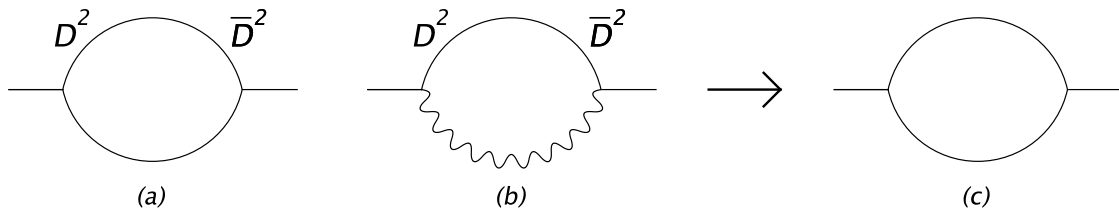
ensures conformal invariance of the theory to all perturbative orders and provides the exact field theory dual to the Lunin-Maldacena supergravity background [3].

The aim of the present investigation is to study how the conformal invariance condition can be implemented for the case of complex  $\beta$ . The analysis is done using a perturbative approach and imposing the finiteness of the two-point chiral correlators. In turn this guarantees the vanishing of all the  $\beta$ -functions [5]. We find that in the planar limit conformal invariance is achieved only for *real* values of the parameter  $\beta$ . This result seems to be in direct correspondence with the findings of the string dual approach in which singular solutions are produced whenever  $\beta$  acquires a non vanishing imaginary part [3, 6–8]. We will comment on this in our conclusions.

In order to perform higher order perturbative calculations it is very efficient to rely on  $\mathcal{N} = 1$  superspace techniques. In this setting the  $\beta$ -deformed theory is described by the following action (we use notations and conventions as in [9], see also [10])

$$\begin{aligned} S = & \int d^8z \operatorname{Tr}(e^{-gV} \bar{\Phi}_i e^{gV} \Phi^i) + \frac{1}{2g^2} \int d^6z \operatorname{Tr}(W^\alpha W_\alpha) \\ & + ih \int d^6z \operatorname{Tr}(q \Phi_1\Phi_2\Phi_3 - q^{-1} \Phi_1\Phi_3\Phi_2) \\ & + i\bar{h} \int d^6\bar{z} \operatorname{Tr}(\bar{q}^{-1} \bar{\Phi}_1\bar{\Phi}_2\bar{\Phi}_3 - \bar{q} \bar{\Phi}_1\bar{\Phi}_3\bar{\Phi}_2) \quad , \quad q \equiv e^{i\pi\beta} \end{aligned} \quad (3)$$

where  $h$  and  $\beta$  are complex couplings and  $g$  is the real gauge coupling constant. The superfield strength  $W_\alpha = i\bar{D}^2(e^{-gV} D_\alpha e^{gV})$  is given in terms of a real prepotential  $V$ , while  $\Phi_i$  with  $i = 1, 2, 3$  are the three chiral superfields of the original  $\mathcal{N} = 4$  SYM



**Figure 1:** Supergraphs contributing at one loop

theory. We write  $V = V^a T_a$ ,  $\Phi_i = \Phi_i^a T_a$  where  $T_a$  are  $SU(N)$  matrices in the fundamental representation. In the undeformed theory one has  $h = g$  and  $q = 1$ .

We want to study the condition that the couplings have to satisfy in order to guarantee the conformal invariance of the theory for complex values of  $h$  and  $\beta$  in the large  $N$  limit. As observed above to this end it is sufficient to impose the finiteness on the two-point chiral correlator [5].

In the large  $N$  limit for real values of  $\beta$ , i.e. if the condition  $q\bar{q} = 1$  is satisfied, the  $\beta$ -deformed theory becomes exactly conformally invariant if the condition (2) is satisfied [4]. This means that if the chiral couplings differ only by a phase from the ones of the  $\mathcal{N} = 4$  SYM theory, the planar limits of the two theories are essentially the same (see also [11]).

When  $q\bar{q} \neq 1$  the easiest way [12] to study the condition of conformal invariance is to look at the difference between the two-point  $\beta$ -deformed correlator and the corresponding one in the  $\mathcal{N} = 4$  SYM theory. If we want to have an exactly marginal deformation the difference must be finite. We will proceed perturbatively in superspace. The propagators for the vector and chiral superfields, and the interaction vertices are obtained directly from the action in (3). Supergraphs are evaluated performing standard  $D$ -algebra in the loops and the corresponding divergent integrals are computed using dimensional regularization in  $D = 4 - 2\epsilon$ .

At one loop the analysis is very simple and mimics exactly what happens in the  $\beta$  real case [13, 14, 12, 4]. The divergent supergraphs are shown in figure 1. The chiral field propagators are given by

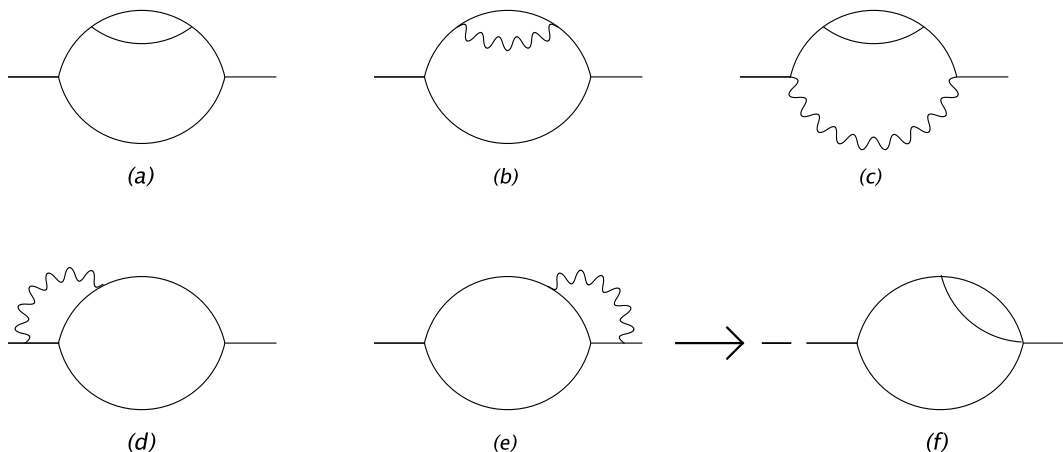
$$\langle \Phi_i \bar{\Phi}_j \rangle = -\delta_{ij} \frac{1}{\square} = \delta_{ij} \frac{1}{p^2} \quad (4)$$

while the vector propagators are

$$\langle VV \rangle = \frac{1}{\square} = -\frac{1}{p^2} \quad (5)$$

The  $D$ -algebra is the same for the two configurations and its completion gives rise to a logarithmically divergent momentum integral. The diagrams (with different color configurations) in figure 1b containing a vector line are the same in the  $\mathcal{N} = 4$  and in the  $\beta$ -deformed SYM theory, since they only depend on the gauge coupling  $g$ . The diagrams in figure 1a contain the chiral couplings: in the deformed theory they give a contribution

$$\frac{N}{(4\pi)^2} h\bar{h} \left( q\bar{q} + \frac{1}{q\bar{q}} \right) \frac{1}{\epsilon} \quad (6)$$



**Figure 2:** Supergraphs contributing at two loops

while in the  $\mathcal{N} = 4$  theory they are proportional to  $g^2$

$$\frac{N}{(4\pi)^2} 2g^2 \frac{1}{\epsilon} \tag{7}$$

In order to achieve finiteness one has to impose that the difference between the two results be finite. This implies that to this order the  $\beta$ -deformed theory is conformal invariant if

$$h\bar{h} \left( q\bar{q} + \frac{1}{q\bar{q}} \right) = 2g^2 \tag{8}$$

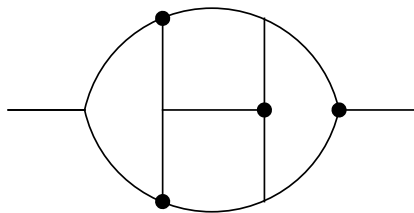
Now we consider higher-loop contributions. Since we look at the difference between the two-point correlators computed in the  $\beta$ -deformed theory and in the  $\mathcal{N} = 4$  SYM, we need not consider diagrams that contain only gauge-type vertices their contributions being the same in the two theories. Therefore we concentrate on divergent graphs that contain either only chiral vertices or mixed chiral and gauge vertices. Moreover we observe that a chiral loop can close only if it has the same number of chiral and antichiral vertices, i.e. no polygonal configurations with an odd number of vertices are possible. With these rules in mind it is straightforward to analyze the two- and three-loop contributions. At two loops we have the diagrams shown in figure 2.

For all the different configurations the D-algebra leads to the same bosonic integral in figure 2f. It is very simple to compute the various color factors: we have for the  $\beta$ -deformed theory

$$\begin{aligned} \text{figure 2a} &\longrightarrow -2 \left[ h\bar{h} \left( q\bar{q} + \frac{1}{q\bar{q}} \right) \right]^2 N^2 \\ \text{figure 2b} + 2c + 2d + 2e &\longrightarrow 2 \left[ h\bar{h} \left( q\bar{q} + \frac{1}{q\bar{q}} \right) \right] g^2 N^2 \end{aligned} \tag{9}$$

while correspondingly for  $\mathcal{N} = 4$  SYM we find

$$\begin{aligned} \text{figure 2a} &\longrightarrow -8g^4 N^2 \\ \text{figure 2b} + 2c + 2d + 2e &\longrightarrow 4g^4 N^2 \end{aligned} \tag{10}$$

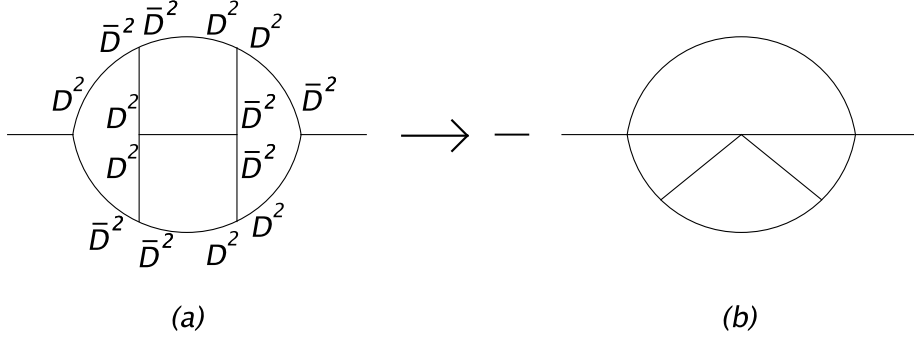


**Figure 3:** New planar structure at four loops; the vertices with dots are antichiral

If we compute the difference of the results in (9) and in (10) and use the conformal invariance condition in (8), we obtain a zero result. This means that the condition we found at one loop ensures finiteness also at two loops. In fact repeating a similar analysis at three loops one can easily show that (8) makes the divergent diagrams computed in the deformed theory equal to the corresponding ones in the  $\mathcal{N} = 4$  SYM. In the planar limit under the condition in (8) the two-point correlators do coincide up to three loops. Up to this order the situation is completely parallel to the case of the real  $\beta$ -deformation [14, 12]: there  $q\bar{q} = 1$  and the condition in (8) was simply given by  $h\bar{h} = g^2$ . This condition was actually sufficient [4] to implement finiteness of the two-point correlator in the planar limit to *all orders* in perturbation theory. Moreover the two-point correlator of the real  $\beta$ -deformed theory becomes *exactly* equal to the one computed in the  $\mathcal{N} = 4$  theory.

Now we proceed in the study of the  $\beta$ -complex case and examine the situation at four loops. We will find that at this order we are forced to modify the condition in (8). This should not come as a surprise because of the following reason: as explained above the divergence at one loop is linked to the color factor of the chiral bubble in figure 1a and this leads to the condition in (8). At two and three loops divergent graphs are constructed either by inserting vector lines on chiral bubbles or by assembling chiral bubbles together. Since the addition of vectors simply modifies the color due to the chiral vertices by the multiplication of  $g^2$  factors, in both cases the condition in (8) suffices to give conformal invariance. In fact this same reasoning applies also to all the four-loop diagrams that either contain vector lines on chiral bubbles or consist of various arrangements of chiral bubbles: for all these cases the condition in (8) makes these graphs equal to the corresponding ones in the  $\mathcal{N} = 4$  theory. The novelty is that at four loops a new type of chiral divergent structure does arise. We will be able to implement the cancelation of divergences at order  $g^8$ , but in contradistinction to the real  $\beta$  case finite parts will survive in the  $\beta$ -deformed two-point function which are absent in the corresponding  $\mathcal{N} = 4$  two-point function.

The new type of chiral supergraph, i.e. not containing chiral bubble insertions, is the one drawn in figure 3. The D-algebra structure shown explicitly in figure 4a is the same for all the arrangements of the three chiral superfields at the vertices. Completing the D-algebra in the loops one obtains the bosonic graph shown in figure 4b. The corresponding



**Figure 4:** D-algebra for the supergraph in figure 3

integral is divergent [15]

$$\begin{aligned}
 I_4 &= - \int \frac{d^D k \, d^D q \, d^D r \, d^D t}{(2\pi)^{4D}} \frac{1}{k^2(k+t)^2(q+k)^2(q+r)^2(q+p)^2 t^2 r^2 (t+r)^2} \\
 &= -5 \zeta(5) \frac{1}{(4\pi)^8} \frac{1}{\epsilon} \frac{1}{(p^2)^{4\epsilon}}
 \end{aligned}
 \tag{11}$$

The color factor is also easily computed: one has to sum over all the various possibilities at the chiral vertices and in so doing one finds

$$C_4 = N^4 (h\bar{h})^4 \left[ (q\bar{q})^4 + \frac{1}{(q\bar{q})^4} + 6 \right]
 \tag{12}$$

The factor in (12) can be rewritten as

$$C_4 = \frac{N^4}{2} (h\bar{h})^4 \left[ \left( q\bar{q} + \frac{1}{q\bar{q}} \right)^4 + \left( q\bar{q} - \frac{1}{q\bar{q}} \right)^4 \right]
 \tag{13}$$

In this way it is easy to compare the result with the one we would have obtained in  $\mathcal{N} = 4$  SYM. In fact using the condition in (8) we find that the  $\beta$ -deformed two-point function at four loops differs from the corresponding  $\mathcal{N} = 4$  two-point function by the contribution

$$J_4 = -\frac{5}{2} \zeta(5) N^4 \frac{1}{(4\pi)^8} \frac{1}{\epsilon} \frac{1}{(p^2)^{4\epsilon}} (h\bar{h})^4 \left( q\bar{q} - \frac{1}{q\bar{q}} \right)^4
 \tag{14}$$

If we want the  $\beta$ -deformed theory to be conformally invariant this term has to be cancelled. The only way out is to modify the relation of conformal invariance in (8), so that a contribution from a lower-loop order might cancel the unwanted four-loop divergence.

In the spirit of [2] (see also [16]), in the space of the coupling constants we are looking for a surface of renormalization group fixed points. To this end we set

$$h_1 \equiv hq \qquad h_2 \equiv \frac{h}{q}
 \tag{15}$$

and reparametrize these couplings in terms of the gauge coupling  $g$ . In fact since in the planar limit for each diagram the color factors from chiral vertices is always in terms of

the products  $h_1^2 \equiv h_1 \bar{h}_1$  and  $h_2^2 \equiv h_2 \bar{h}_2$  we express directly  $h_1^2$  and  $h_2^2$  as power series in the coupling  $g^2$  as follows

$$\begin{aligned} h_1^2 &= a_1 g^2 + a_2 g^4 + a_3 g^6 + \dots \\ h_2^2 &= b_1 g^2 + b_2 g^4 + b_3 g^6 + \dots \end{aligned} \tag{16}$$

The coefficients  $a_i$  and  $b_i$  will be determined by imposing that what we obtain from various loop orders, subtracted by the corresponding  $\mathcal{N} = 4$  results, vanishes order by order in the  $g^2$  expansion.

In order to make the comparison with the  $\mathcal{N} = 4$  calculation simpler we find convenient to determine the general structure of the color factors of the relevant diagrams. At  $L$ -loop order the color factor is a homogeneous polynomial in  $h_1^2, h_2^2$  and  $g^2$  of degree  $L$ . Moreover, as a consequence of the invariance of the theory under the global symmetry  $h_1 \leftrightarrow -h_2$  and  $\Phi_i \leftrightarrow \Phi_j, i \neq j$ , it has to be symmetric under  $h_1^2 \leftrightarrow h_2^2$ . These properties, together with the requirement of having a smooth limit to  $(2g^2)^L$  in the  $\mathcal{N} = 4$  limit ( $h_1^2, h_2^2 \rightarrow g^2$ ), constrain the  $L$ -loop color factor to have the following form <sup>1</sup>

$$F^{(L)}(h_1^2 + h_2^2) + (h_1^2 - h_2^2)^2 G^{(L-2)}(h_1^2, h_2^2) \tag{17}$$

with  $F^{(L)}(2g^2) = (2g^2)^L$ . The functions  $F^{(L)}$  and  $G^{(L-2)}$  depend also on the coupling  $g^2$ , but for notational simplicity we have chosen not to write it explicitly. They are homogeneous polynomials of degrees  $L$  and  $(L-2)$  respectively, symmetric in  $h_1^2, h_2^2$ . Their general form is

$$\begin{aligned} F^{(L)}(h_1^2 + h_2^2) &= \sum_{k=0}^L (h_1^2 + h_2^2)^k (2g^2)^{L-k} f_k \\ G^{(L-2)}(h_1^2, h_2^2) &= \sum_{k=0}^{[(L-2)/2]} (h_1^2 - h_2^2)^{2k} \mathcal{P}^{(L-2-2k)}(h_1^2, h_2^2) \end{aligned} \tag{18}$$

with constant coefficients  $f_k$  satisfying  $\sum_{k=0}^L f_k = 1$  and  $\mathcal{P}^{(L-2-2k)}$  homogeneous polynomials not vanishing for  $h_1^2 = h_2^2$ .

We note that for pure chiral diagrams, the ones we will be mainly interested in, there is no  $g^2$ -dependence in  $F^{(L)}$  and  $G^{(L-2)}$  and, in particular,  $F^{(L)}(h_1^2 + h_2^2) = (h_1^2 + h_2^2)^L$ .

At  $L$ -loop order, after we take the difference with the  $\mathcal{N} = 4$  result what is left over is given by

$$\Gamma^{(L)} = \left[ F^{(L)}(h_1^2 + h_2^2) - (2g^2)^L + (h_1^2 - h_2^2)^2 G^{(L-2)}(h_1^2, h_2^2) \right] I_{\text{div}}^{(L)} \tag{19}$$

where  $I_{\text{div}}^{(L)}$  denotes the divergent factor from the  $L$ -loop integral. Finally summing over all loops and using the expansions in (16) we end up with

$$\begin{aligned} \sum_L \Gamma^{(L)} &= \sum_L \left[ F^{(L)}(h_1^2 + h_2^2) - (2g^2)^L + (h_1^2 - h_2^2)^2 G^{(L-2)}(h_1^2, h_2^2) \right] I_{\text{div}}^{(L)} \\ &= \sum_k A_k (g^2)^k \end{aligned} \tag{20}$$

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<sup>1</sup>We do not worry about an overall normalization factor since it is irrelevant for our general argument

Conformal invariance is achieved imposing

$$A_k = 0 \tag{21}$$

order by order in  $g^2$ .

Thus we go back to the one-loop calculation and apply concretely the general procedure described above. From the results quoted in (6) and (7) we see that  $G^{(-1)} = 0$  and find

$$\Gamma^{(1)} = \left[ F^{(1)}(h_1^2 + h_2^2) - (2g^2) \right] I_{\text{div}}^{(1)} = \frac{N}{(4\pi)^2} [h_1^2 + h_2^2 - 2g^2] \frac{1}{\epsilon} \tag{22}$$

Therefore using the expansions in (16) at order  $g^2$  we have to impose the condition

$$\mathcal{O}(g^2) : \quad A_1 = 0 \quad \longrightarrow \quad a_1 + b_1 - 2 = 0 \tag{23}$$

In fact since we have shown that the condition in (8) ensures conformal invariance up to three loops, up to order  $g^6$ , we find the following additional requirements

$$\begin{aligned} \mathcal{O}(g^4) : \quad & A_2 = 0 \quad \longrightarrow \quad a_2 + b_2 = 0 \\ \mathcal{O}(g^6) : \quad & A_3 = 0 \quad \longrightarrow \quad a_3 + b_3 = 0 \end{aligned} \tag{24}$$

At this point it should be clear that, according to the procedure we have illustrated above, we do not need consider anymore diagrams containing insertions of chiral bubbles like the one in figure 1a: once the condition (23) is satisfied these diagrams do not lead to new divergent contributions. Therefore at every loop order we have to isolate diagrams corresponding to new chiral structures with eventually vector propagators inserted on them.

Now we reexamine the results we have obtained up to four loops, i.e. up to order  $g^8$ . From the four-loop calculation (see eqs. (11) and (13)) we have

$$\begin{aligned} & -\frac{5}{2} \zeta(5) N^4 \frac{1}{(4\pi)^8} \frac{1}{\epsilon} (h\bar{h})^4 \left\{ \left( q\bar{q} + \frac{1}{q\bar{q}} \right)^4 + \left( q\bar{q} - \frac{1}{q\bar{q}} \right)^4 \right\} \\ & = -\frac{5}{2} \zeta(5) N^4 \frac{1}{(4\pi)^8} \frac{1}{\epsilon} [(h_1^2 + h_2^2)^4 + (h_1^2 - h_2^2)^4] \end{aligned} \tag{25}$$

Therefore we find

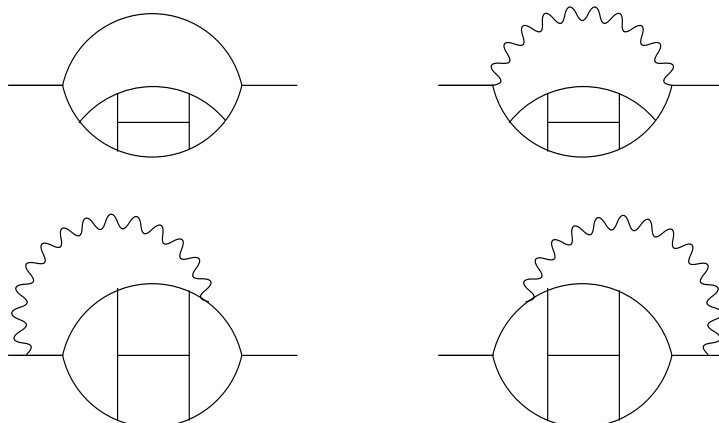
$$\Gamma^{(4)} = -\frac{5}{2} \zeta(5) N^4 \frac{1}{(4\pi)^8} \frac{1}{\epsilon} [(h_1^2 + h_2^2)^4 - (2g^2)^4 + (h_1^2 - h_2^2)^4] \tag{26}$$

Now we insert into (20) the results we have found so far, i.e. (22) and (26) and use the expansions in (16) with the conditions in (23) and (24). In this way we find that the conformal invariance condition at order  $g^8$  is satisfied if

$$\mathcal{O}(g^8) : \quad A_4 = 0 \quad \longrightarrow \quad a_4 + b_4 - \frac{5}{2} \zeta(5) N^3 \frac{1}{(4\pi)^6} (a_1 - b_1)^4 = 0 \tag{27}$$

Up to this point we have ensured that the two-point function is finite up to the order  $g^8$ . The finite contributions explicitly depend on  $q$  and vanish in the corresponding terms of the  $\mathcal{N} = 4$  theory.





**Figure 5:** Planar supergraphs with  $1/\epsilon^2$  divergences at five loops

The next step leads us to order  $g^{10}$ : we have to consider the new five-loop diagrams and the two-loop diagrams that will talk to the five-loop graphs once the conformal invariance condition (27) is imposed. Following the procedure described so far, i.e. implementing the conformal invariance condition order by order in the couplings, at the order  $g^8$  we ended up adding contributions coming from one-loop integrals and from four-loop integrals. Now these structures show up at order  $g^{10}$  as subdivergences in two-loop and five-loop integrals respectively and they are responsible for the insurgence of  $1/\epsilon^2$ -pole terms. In figure 2 and in figure 5 we have drawn the two- and five-loop diagrams which give rise to  $1/\epsilon^2$ -pole terms. Having cancelled divergences at lower orders one might be tempted to believe that these  $1/\epsilon^2$  terms would automatically add up to zero. Indeed this would be the case if we were cancelling divergences order by order in loops. As emphasized above we are proceeding order by order in the coupling  $g^2$ . At the order  $g^8$  imposing the relation (27) we have cancelled the  $1/\epsilon$  pole from the one-loop diagram in figure 1c with the  $1/\epsilon$  pole appearing from the graph at four loops in figure 4b. Essentially if we write schematically the one-loop result as

$$A \frac{1}{\epsilon} \frac{1}{(p^2)^\epsilon} \tag{28}$$

and the four-loop result as

$$B \frac{1}{\epsilon} \frac{1}{(p^2)^{4\epsilon}} \tag{29}$$

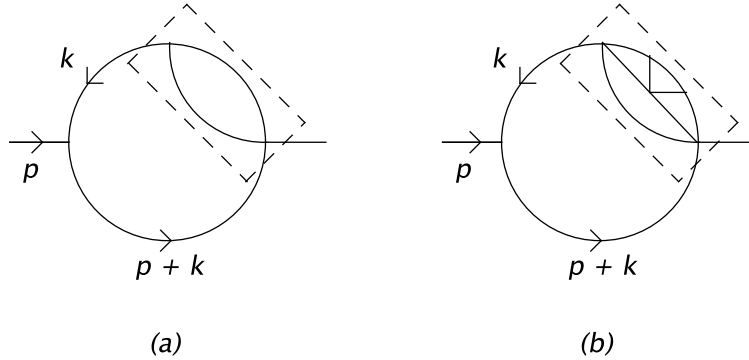
imposing the relation in (27) we have set  $A + B = 0$ . When we go one loop higher we have to deal with the bosonic integrals shown in figure 6.

The  $1/\epsilon^2$  term in figure 6a arises from

$$A \frac{1}{\epsilon} \int d^D k \frac{1}{(p+k)^2 (k^2)^{1+\epsilon}} \longrightarrow A \frac{1}{\epsilon} \Gamma(2\epsilon) \tag{30}$$

The  $1/\epsilon^2$  term in figure 6b arises from

$$B \frac{1}{\epsilon} \int d^D k \frac{1}{(p+k)^2 (k^2)^{1+4\epsilon}} \longrightarrow B \frac{1}{\epsilon} \Gamma(5\epsilon) \tag{31}$$



**Figure 6:** Subtraction of subdivergences at order  $g^{10}$

It is clear that setting  $A + B = 0$  is not enough to cancel the  $1/\epsilon^2$  poles.

In order to check this general argument we have computed the  $1/\epsilon^2$  divergent terms explicitly. At order  $g^{10}$  from the two-loop graphs shown in figure 2, denoting with  $I_2$  the divergent integral in figure 6a we have

$$-6(a_4 + b_4)N^2 I_2 \quad \longrightarrow \quad -15 \zeta(5) N^5 \frac{1}{(4\pi)^6} (a_1 - b_1)^4 \frac{1}{(4\pi)^4} \frac{1}{2\epsilon^2} \quad (32)$$

where we have used the relation in (27). In the same way from the five-loop graphs shown in figure 5, denoting with  $I_5$  the divergent integral in figure 6b, we obtain

$$3(a_1 - b_1)^4 N^5 I_5 \quad \longrightarrow \quad 3(a_1 - b_1)^4 N^5 \frac{1}{(4\pi)^{10}} \frac{\zeta(5)}{\epsilon^2} \quad (33)$$

Clearly the terms in (32) and (33) do not add up to zero and in fact they reproduce the mismatch anticipated in (30) and (31) when  $A + B = 0$ . Therefore at order  $g^{10}$  the cancellation of the  $1/\epsilon^2$  poles requires that (see also (23) and (27))

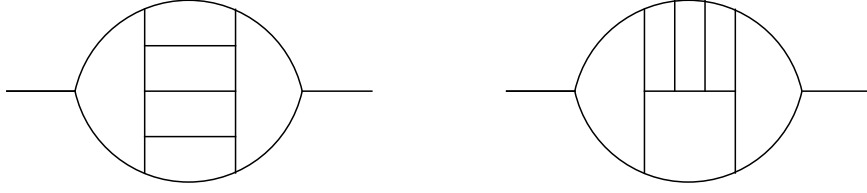
$$a_1 = b_1 = 1 \quad a_4 + b_4 = 0 \quad (34)$$

These conditions can be equivalently found by requiring a vanishing gauge  $\beta$ -function at order  $g^{11}$  [17].

Once the conditions in (34) have been imposed, at the order  $g^{10}$  all the  $1/\epsilon$  divergences from diagrams at five and two loops are automatically cancelled. Thus at this order the only divergence comes from the one-loop bubble and we are forced to impose

$$a_5 + b_5 = 0 \quad (35)$$

Before proceeding to the next order  $g^{12}$ , let us note that this pattern of cancelling divergences between the one-loop bubble in figure 1a and the four-loop diagram in figure 4 will repeat itself at order  $g^{16}$ , while the cancelation of the  $1/\epsilon^2$  poles will show up at the order  $g^{18}$  and will involve again the diagrams at two and five loops that we have just considered. Indeed at this stage from the divergent contribution of the four-loop diagrams,



**Figure 7:** New planar chiral diagrams at six loops

using the conditions imposed so far on the coefficients of the expansions in (16), the first divergence will be proportional to

$$[(a_2 - b_2) g^4]^4 = (2a_2)^4 g^{16} \quad (36)$$

So for the time being, having ensured conformal invariance of the theory up to the order  $g^{10}$ , we proceed and examine the situation at six loops. The new divergent chiral diagrams are shown in figure 7: they are all logarithmically divergent.

Their color factor is easily evaluated: it can be written in the following form

$$(h_1^2 + h_2^2)^6 + (h_1^2 - h_2^2)^4 \left( \frac{5}{3} h_1^4 + \frac{2}{3} h_1^2 h_2^2 + \frac{5}{3} h_2^4 \right) \quad (37)$$

Thus we find that in the  $g^2$  expansion the first nonvanishing term from the six-loop divergence will be proportional to

$$[(a_2 - b_2) g^4]^4 g^4 = (2a_2)^4 g^{20} \quad (38)$$

Thus once again to the order  $g^{12}$  the only divergence arises from the one-loop bubble and its cancelation requires

$$a_6 + b_6 = 0 \quad (39)$$

We keep on going and look for divergent terms at the order  $g^{14}$ . The diagrams at seven loops have a color factor proportional to

$$(h_1^2 + h_2^2)^7 + (h_1^2 - h_2^2)^4 (3h_1^6 + 5h_1^4 h_2^2 + 5h_1^2 h_2^4 + 3h_2^6) \quad (40)$$

which, using the expansion in (16), gives as first relevant term

$$[(a_2 - b_2) g^4]^4 g^6 = (2a_2)^4 g^{22} \quad (41)$$

Therefore once again the only divergence at the order  $g^{14}$  comes from one loop and leads to the condition

$$a_7 + b_7 = 0 \quad (42)$$

In accordance with the general discussion around equations (17), (18) and what we have found by the explicit calculations we have reported up to seven loops, we write the  $L$ -loop color structure of the pure chiral divergent diagrams in the following form

$$(h_1^2 + h_2^2)^L + (h_1^2 - h_2^2)^4 [\alpha (h_1^2)^{L-4} + \beta (h_1^2)^{L-5} h_2^2 + \dots + \gamma (h_2^2)^{L-4}] \quad (43)$$

We note that the only arbitrary assumption with respect to the general form that one can infer from (17), (18) is the absence of a term proportional to  $(h_1^2 - h_2^2)^2$ . Even if we do not have a general argument for the absence of such a term we are very well supported by the results up to seven loops illustrated so far.

If we take into account the conditions found so far for the coefficients in (16), then (43) immediately implies that the various diagrams at  $L$  loops will give contributions in the  $g^2$  expansion whose first relevant term is proportional to

$$[(a_2 - b_2) g^4]^4 g^{2L-8} = (2a_2)^4 g^{2L+8} \quad (44)$$

The conclusion is that diagrams at six loops or higher will start contributing at the earliest when we reach order  $g^{20}$ , as we have explicitly seen in (38) and (41). Therefore if we now turn to the order  $g^{16}$ , as previously anticipated, the only divergent contributions come from the one-loop bubble proportional to  $a_8 + b_8$  and from the four-loop diagram proportional to  $a_2^4$  (see eq. (36)). In order for the divergences to cancel at this order we have to require

$$\mathcal{O}(g^{16}) : \quad A_8 = 0 \quad \longrightarrow \quad a_8 + b_8 - \frac{5}{2} \zeta(5) N^3 \frac{1}{(4\pi)^6} (a_2 - b_2)^4 = 0 \quad (45)$$

Going up to the order  $g^{18}$  we have to cancel the  $1/\epsilon^2$  poles from the two and five-loop diagrams: following the same steps as before we are forced to impose

$$a_8 + b_8 = 0 \quad a_2 = b_2 = 0 \quad (46)$$

With these conditions on the coefficients in the expansion (16), at order  $g^{18}$  the  $1/\epsilon$  poles come only from the one-loop bubble and they cancel out once

$$a_9 + b_9 = 0 \quad (47)$$

Since in (46) we have imposed  $a_2 = 0$ , automatically we find that the various divergences from six, seven,  $\dots$ ,  $L$ -loop diagrams are pushed up

$$\begin{aligned} 6 \text{ loops} &\longrightarrow [(a_3 - b_3) g^6]^4 g^4 = (2a_3)^4 g^{28} \\ 7 \text{ loops} &\longrightarrow [(a_3 - b_3) g^6]^4 g^6 = (2a_3)^4 g^{30} \\ \dots &\dots \dots \\ L \text{ loops} &\longrightarrow [(a_3 - b_3) g^6]^4 g^{2L-8} = (2a_3)^4 g^{2L+16} \end{aligned} \quad (48)$$

It becomes clear that everything is ruled by the cancelation of divergences at one and four loop and by the subsequent cancelation of the  $1/\epsilon^2$  poles at two and five loops. This happens at the order  $(g^2)^{4k}$  and at the order  $(g^2)^{4k+1}$  respectively. The new chiral graphs at six loops and higher never enter the game due to the specific form of their color structure as in (43). The mechanism works as follows: up to the order  $(g^2)^{4k-1}$  we find that the coefficients have to satisfy

$$a_1 = b_1 = 1 \quad a_{j-1} = 0 \quad a_{4j-1} + b_{4j-1} = 0 \quad j = 2, \dots, k \quad (49)$$

At  $\mathcal{O}((g^2)^{4k})$  in order to cancel the divergent contributions from one and four loops we have to impose

$$a_{4k} + b_{4k} - \frac{5}{2} \zeta(5) N^3 \frac{1}{(4\pi)^6} (a_k - b_k)^4 = 0 \quad (50)$$

Then at  $\mathcal{O}((g^2)^{4k+1})$  the divergences from two and five loops need to be cancelled and we are forced to require

$$a_{4k} + b_{4k} = 0 \quad a_k = b_k = 0 \quad (51)$$

Finally this leads to

$$a_1 = b_1 = 1 \quad a_n = b_n = 0 \quad n = 2, 3, \dots \quad (52)$$

These conclusions have been drawn based on the general expression given in (43) for the color structure of pure chiral diagrams where we have assumed the absence of a term quadratic in  $(h_1^2 - h_2^2)$ . Now which control do we have on this assumption in the higher-loop divergent chiral diagrams? We have computed explicitly all the color structures up to ten loops; with the help of Mathematica we have evaluated the color factors of arbitrarily chosen higher-loop graphs; in addition we have explicit formulas for several classes of chiral diagrams. We have found consistently that all of them can be cast in the form given in (43).

The conditions (52) on the coefficients tell us that the  $\beta$ -deformed SYM theory is conformally invariant only for  $\beta$  real.

In the AdS/CFT dual description supergravity solutions associated to a complex parameter can be generated by completing the usual TsT transformation which leads to the Lunin-Maldacena background with S-duality transformations [3]. However, as discussed in [6], S-duality transformations might affect the 2d conformal invariance of the string sigma-model and this would require the appearance of  $\alpha'/R^2$  corrections to the classical superstring action and then to the Lunin-Maldacena background. The fact that a complex deformation parameter might be problematic is also signaled by the appearance of singularities in the deformed metric when an imaginary part of  $\beta$  is turned on [7]. Therefore, the result we have obtained on the field theory side seems to be in agreement with AdS/CFT expectations.

Our result is also consistent with the integrability properties of the deformed theories. In fact, the requirement that the one-loop dilatation operator of a generic deformed theory (3) be the hamiltonian of an integrable spin chain constrains  $|q| = 1$  [18].

We stress that our investigation has been carried on perturbatively, ignoring completely possible nonperturbative effects. In particular, we have assumed the gauge coupling constant to be real in order to avoid the presence of nontrivial instantonic effects [19]. It would be interesting to extend our analysis to  $g$  complex and to understand if the embedding of all the couplings in a complex manifold leads to nontrivial superconformal conditions.

## Acknowledgments

This work has been supported in part by INFN, PRIN prot.2005024045-002 and the European Commission RTN program MRTN-CT-2004-005104.

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